Instructions: Try these problems from the textbook as practice (answers below).

Problems

4-7. Compute the directional derivative of f at the given point in the direction indicated by the angle θ.

5. $f(x,y) = y\cos(xy),\ (0,1),\ \theta = \pi/4$
7. $f(x,y) = \arctan(xy),\ (2,-3),\ \theta = 3\pi/4$

8-12. For the following functions, compute $\nabla f(P)$ and the rate of change at P in the direction of the vector $\mathbf{u}$.

9. $f(x,y) = x/y,\ P(3,0),\ \mathbf{u} = \left\langle \frac{3}{5}, \frac{4}{5} \right\rangle$
11. $f(x,y,z) = x^2yz - xyz^2,\ P(2,-1,1),\ \mathbf{u} = \left\langle 0, \frac{4}{5}, -\frac{3}{5} \right\rangle$

13-19. Compute the directional derivative of the function at the given point in the direction of the vector $\mathbf{v}$.

13. $f(x,y) = e^x \sin y,\ (0,\pi/3),\ \mathbf{v} = \langle -6, 8\rangle$
15. $g(s,t) = s\sqrt{t},\ (2,4),\ \mathbf{v} = \langle 2,-1\rangle$
17. $f(x,y,z) = x^2y + y^2z,\ (1,2,3),\ \mathbf{v} = \langle 2,-1,2\rangle$
19. $h(r,s,t) = \ln(3r+6s+9t),\ (1,1,1),\ \mathbf{v} = \langle 4,12,6\rangle$

21-25. Compute the directional derivative of the function at the point P in the direction of the point Q.

21. $f(x,y) = x^2y^2 - y^3,\ P(1,2),\ Q(-3,5)$
23. $f(x,y) = \sqrt{xy},\ P(2,8),\ Q(5,4)$
25. $f(x,y,z) = xy - xy^2z^2,\ P(2,-1,1),\ Q(5,1,7)$

27-32. Compute the maximum rate of change of f at the given point and the direction in which it occurs.

27. $f(x,y) = 5xy^2,\ (3,-2)$
29. $f(x,y) = \sin(xy),\ (1,0)$
31. $f(x,y,z) = x/(y+z),\ (8,1,3)$

35. Find all points at which the direction of fastest change of the function $f(x,y) = x^2 + y^2 - 2x - 4y$ is $\mathbf{i} + \mathbf{j}$.

37. The temperature T at the points in a metal ball centred at the origin is inversely proportional to its distance from the origin. The temperature at the point $(1,2,2)$ is $120°$.

(a) Find the rate of change of T at $(1,2,2)$ in the direction toward the point $(2,1,3)$.
(b) Show that at every point in the ball the direction of greatest increase in temperature is given by a vector that points to the origin.

47-52. Find equations of the tangent plane to the given surface at the specified point.

47. $2(x-2)^2 + (y-1)^2 + (z-3)^2 = 10,\ (3,3,5)$
49. $xy^2z^3 = 8,\ (2,2,1)$
51. $x + y + z = e^{xyz},\ (0,0,1)$

55. If $f(x,y) = xy$, find the gradient vector $\nabla f(3,2)$ and use it to find the tangent line to the level $f(x,y) = 6$ at the point $(3,2)$. Sketch the level curve, the tangent line, and the gradient vector.

61. Are there any points of the surface $x^2 - y^2 - z^2 = 1$ where the tangent plane is parallel to the plane $z = x + y$?

63. Show that every plane that is tangent to the cone $x^2 + y^2 = z^2$ passes through the origin.

69. Find parametric equations for the tangent line to the curve of intersection of the paraboloid $z = x^2 + y^2$ and the ellipsoid $4x^2 + y^2 + z^2 = 9$ at the point $(-1,1,2)$.

71. Where does the helix $\mathbf{r}(t) = \langle \cos(\pi t), \sin(\pi t), t\rangle$ intersect the paraboloid $z = x^2 + y^2$? What is the angle of intersection between the helix and the paraboloid?

Answers

5. $D_{\mathbf{u}}f(P) = \sqrt{\frac{1}{2}}$
7. $D_{\mathbf{u}}f(P) = \dfrac{5}{37\sqrt{2}}$
9. f is not continuous at the given point
11. $D_{\mathbf{u}}f(P) = \frac{8}{5}$
13. $D_{\mathbf{u}}f(P) = \dfrac{4-3\sqrt{3}}{10}$
15. $D_{\mathbf{u}}g(P) = \dfrac{7}{2\sqrt{5}}$
17. $D_{\mathbf{u}}f(P) = 1$
19. $D_{\mathbf{u}}h(P) = \dfrac{23}{42}$
21. $D_{\mathbf{u}}f(P) = -\dfrac{56}{5}$
23. $D_{\mathbf{u}}f(P) = 2/5$
25. $D_{\mathbf{u}}f(P) = -\dfrac{22}{7}$
27. $\nabla f(P) = \langle 20, -60\rangle$, so $|\nabla f(P)| = 20\sqrt{10}$
29. $\nabla f(P) = \langle 1, 0\rangle$, so $|\nabla f(P)| = 1$
31. $\nabla f(P) = \left\langle \dfrac{1}{4}, -\dfrac{1}{2}, -\dfrac{1}{2}\right\rangle$, so $|\nabla f(P)| = \dfrac{3}{4}$
35. $(\frac{3}{2}, \frac{5}{2})$

37a. $D_{\mathbf{u}}T(P) = \dfrac{60}{\sqrt{3}}$

47. $x + y + z = 11$
49. $(x-2) + 2(y-2) + 6(z-1) = 0$
51. $z = 1$
55. $\nabla f(3, 2) = \langle 2, 3\rangle$
61. No.
69. $x = t - 1$, $y = 1$, $z = 2t + 2$
71. $(-1, 0, 1)$. The normal of the tangent plane to $z = x^2 + y^2$ is at an angle of $\arccos(-1/\sqrt{5 + 5\pi^2})$ with the tangent vector to the helix at that point.